

# A Bernstein type theorem for graphic self-shrinkers with flat normal bundle

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## Abstract

In this note we will prove that an  $n$  dimensional graphic self-shrinker in  $R^{n+m}$  with flat normal bundle is a linear subspace. This result is a generalization of the corresponding result of Lu Wang in codimension one case.

## 1 Introduction

Let  $\{M_t\}_{T>t>0}$  be a family of  $n$  dimensional submanifolds in  $R^{n+m}$  and let  $H_t$  be the mean curvature vector of  $M_t$  in  $R^{n+m}$ , then  $\{M_t\}_{T>t>0}$  is said to be moving by mean curvature if

$$\frac{d}{dt}M_t = H_t. \quad (1.1)$$

Let  $M$  be a  $n$  dimensional submanifold in  $R^{n+m}$  and  $NM$  be its normal bundle.  $M$  is said to be a self-shrinker if it satisfies

$$H = -\frac{1}{2}\vec{x}^N, \quad (1.2)$$

where  $\vec{x}$  is the position vector of  $R^{n+m}$  and  $\vec{x}^N$  is the normal component of  $\vec{x}$  in  $NM$ .

Self-shrinkers are the simplest solutions to the mean curvature flow and they are important in the singularity analysis of the mean curvature flow, see for example [CM2], [CM3], [Hui1] and [Hui2].

On the other hand, self-shrinkers can be viewed as minimal surfaces endowed with the weighted metric  $e^{-\frac{|\vec{x}|^2}{2n}}\delta_{ij}$ , see [Ang], [CM2] and [CM3]. In the theory of minimal surfaces the Bernstein theorem for entire minimal graphs played a fundamental role (see, for example [CM1] and [Oss]), and so it is nature to ask whether there are Bernstein type theorems for graphic self-shrinkers. Ecker and Huisken proved that  $n$  dimensional smooth self-shrinkers in  $R^{n+1}$ , which are entire graphs and have at most polynomial volume growth, are hyperplanes, in the appendix of [EH]. The volume growth condition is removed by Wang [W]. She proved that an  $n$  dimensional self-shrinker of entire graph in  $R^{n+1}$  has polynomial volume growth by using a calibration argument and a  $L^\infty$  estimate to the graph function.

In this note, we will prove an Bernstein type theorem for graphic self-shrinkers with flat normal bundles of arbitrary codimension, which generalizes the result of Lu Wang. The main step is to get a stability inequality for the self-shrinker with the weighted measure  $e^{-\frac{|\vec{x}|^2}{4}}d\mu$ , by combining the ideas from [Xin] and [W]. Our theorem is stated as follows.

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**Theorem 1.1.** *Let  $M$  be an  $n$  dimensional graphic self-shrinker in  $R^{n+m}$  with flat normal bundle, then  $M$  is a linear subspace.*

Comparing to the proof of the Bernstein theorem for minimal graphs with flat normal bundle in [SWX], our proof of the above theorem is quite simple. The reason is that in our case we can get a "weighted stability inequality" (Lemma 3.1) with weight  $e^{-\frac{|\vec{x}|^2}{4}}$  for self-shrinkers, and then by the volume growth estimate of Ding and Xin (Theorem 2.1) the right hand side of the weighted stability inequality tends to zero by choosing appropriate cut off functions.

For other Bernstein type theorems for graphic self-shrinkers, we refer to [CCY], [DW], [DX2], [DXY] and [HW].

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## 2 Preliminaries

### 2.1 Basic equations from submanifold theory

Let  $M \hookrightarrow \overline{M}$  be an isometric immersion with the second fundamental form  $B$ , which can be viewed as a section of the vector bundle  $Hom(\odot^2 TM, NM)$  over  $M$ , where  $TM$  and  $NM$  are the tangent bundle and normal bundle of  $M$ , respectively. Then the second fundamental form, the curvature tensor of the submanifolds, curvature tensor of the normal bundle and that of the ambient manifold, satisfy the Gauss equation, the Codazzi equation, and the Ricci equation, as follows.

$$\begin{aligned}\langle R_{XY}Z, W \rangle &= \langle \overline{R}_{XY}Z, W \rangle - \langle B_{XW}, B_{YZ} \rangle + \langle B_{XZ}, B_{YW} \rangle, \\ (\nabla_X B)_{YZ} &- (\nabla_Y B)_{XZ} = -(\overline{R}_{XY}Z)^N, \\ \langle R_{XY}\mu, \nu \rangle &= \langle \overline{R}_{XY}\mu, \nu \rangle - \langle B_{Xe_i}, \mu \rangle \langle B_{Ye_i}, \nu \rangle - \langle B_{Xe_i}, \nu \rangle \langle B_{Ye_i}, \mu \rangle,\end{aligned}$$

where  $\{e_i\}$  is a local orthonormal frame on  $M$ ;  $X, Y, Z \in TM$  and  $\mu, \nu \in NM$ .

In particular if  $\overline{M}$  is the Euclidean space and  $M$  has flat normal bundle, then the Ricci equation will be

$$\langle B_{Xe_i}, \mu \rangle \langle B_{Ye_i}, \nu \rangle = \langle B_{Xe_i}, \nu \rangle \langle B_{Ye_i}, \mu \rangle. \quad (2.1)$$

### 2.2 Volume growth of self-shrinkers

The following theorem estimates the volume growth of self-shrinkers with any codimension ([DX1]).

**Theorem 2.1** (Ding-Xin). *Any complete non-compact properly immersed  $n$  dimensional self-shrinker  $M$  in  $R^{n+m}$  has Euclidean volume growth at most. Precisely,  $\int_{D_r} d\mu \leq Cr^n$  for  $r \geq 1$ , where  $C$  is a constant depending on  $n$  and the volume of  $D_{8n}$ , and  $D_r = M \cap B_r$ ,  $B_r$  is the ball of  $R^{n+m}$  centered at the origin with radius  $r$ .*

It is proved in [CZ] that the inverse is also true. That is any complete self-shrinker in  $R^{n+m}$  with Euclidean volume growth must be proper.

### 3 Proof of theorem 1.1

For vectors  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  in  $R^{n+m}$ , we let  $A = a_1 \wedge \dots \wedge a_n$  and  $B = b_1 \wedge \dots \wedge b_n$  and define their inner product by

$$\langle A, B \rangle = \det(\langle a_i, b_j \rangle).$$

Let  $M$  be an  $n$  dimensional complete self-shrinker of  $R^{n+m}$  with flat normal bundle. For a point  $x \in M$  we chose an orthonormal frame field  $\{e_i, e_\alpha\}$  such that  $e_i \in TM$  and  $e_\alpha \in NM$ . Fix a  $n$ -vector  $A = a_1 \wedge \dots \wedge a_n$ . We define a function on  $M$  by

$$\omega = \langle e_1 \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle = \det(\langle e_i, a_j \rangle). \quad (3.1)$$

Then we have

$$\begin{aligned} e_i(\omega) &= \sum_j \langle e_1 \wedge \dots \wedge D_{e_i} e_j \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \\ &= \sum_j \langle e_1 \wedge \dots \wedge (D_{e_i} e_j)^T \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \\ &+ \sum_j \langle e_1 \wedge \dots \wedge (D_{e_i} e_j)^N \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \\ &= \sum_{\alpha, j} h_{\alpha i j} \langle e_1 \wedge \dots \wedge e_\alpha \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \end{aligned}$$

and so

$$\begin{aligned} \Delta \omega &= -|B|^2 \omega + \sum_{\alpha, i, j} h_{\alpha i j} \langle e_1 \wedge \dots \wedge e_\alpha \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \\ &+ \sum_{\alpha, \beta, i, j, k} \langle e_1 \wedge \dots \wedge h_{\alpha i j} e_\alpha \wedge \dots \wedge h_{\beta i k} e_\beta \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \\ &= -|B|^2 \omega + \sum_{\alpha, i, j} h_{\alpha i j} \langle e_1 \wedge \dots \wedge e_\alpha \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle \\ &+ \sum_{\alpha < \beta, i, j, k} (h_{\alpha i j} h_{\beta i k} - h_{\beta i j} h_{\alpha i k}) \langle e_1 \wedge \dots \wedge e_\alpha \wedge \dots \wedge e_\beta \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle. \end{aligned}$$

Because  $M$  is a submanifold in  $R^{n+m}$  with flat normal bundle, by (2.1) we have

$$\sum_{\alpha < \beta, i, j, k} (h_{\alpha i j} h_{\beta i k} - h_{\beta i j} h_{\alpha i k}) = 0$$

and therefore we have

$$\Delta \omega = -|B|^2 \omega + \sum_{\alpha, i, j} h_{\alpha i j} \langle e_1 \wedge \dots \wedge e_\alpha \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle.$$

On the other hand,

$$H = -\frac{1}{2} \vec{x}^N = -\frac{1}{2} \sum_{\alpha} \langle \vec{x}, e_\alpha \rangle e_\alpha,$$

and so

$$\sum_{\alpha,i,j} h_{\alpha i j} = \sum_{\alpha,j} e_j \langle H, e_\alpha \rangle = - \sum_{\alpha,j} \frac{1}{2} e_j \langle \vec{x}, e_\alpha \rangle = \sum_{\alpha,i,j} \frac{1}{2} h_{\alpha j i} \langle \vec{x}, e_i \rangle,$$

which implies that

$$\sum_{\alpha,i,j} h_{\alpha i j} \langle e_1 \wedge \dots \wedge e_\alpha \wedge \dots \wedge e_n, a_1 \wedge \dots \wedge a_n \rangle = \frac{1}{2} \sum_i \langle (e_i \omega) e_i, \vec{x} \rangle = \frac{1}{2} \langle \nabla \omega, \vec{x} \rangle.$$

Finally we obtain the following formula for  $\omega$

$$\Delta \omega - \frac{1}{2} \langle \nabla \omega, \vec{x} \rangle + |B|^2 \omega = 0. \quad (3.2)$$

If  $\omega > 0$  on  $M$ , then let  $g = \log \omega$  and  $g$  satisfies the following equation

$$\Delta g + |\nabla g|^2 - \frac{1}{2} \langle \nabla g, \vec{x} \rangle + |B|^2 = 0.$$

We have the following weighted stability inequality

**Lemma 3.1.** *Let  $M$  be an  $n$  dimensional complete self-shrinker in  $R^{n+m}$  with flat normal bundle. If there is an  $n$ -vector  $A$  such that the function  $\omega$  defined by (3.1) is positive everywhere on  $M$ , then*

$$\int_M |B|^2 \eta^2 e^{-\frac{|\vec{x}|^2}{4}} d\mu \leq \int_M |\nabla \eta|^2 e^{-\frac{|\vec{x}|^2}{4}} d\mu,$$

where  $\eta$  is any function with compact support on  $M$ .

*Proof.* Multiplying the equation of  $g$  by  $\eta^2 e^{-\frac{|\vec{x}|^2}{4}}$  and integrating over  $M$  gives

$$\begin{aligned} 0 &= \int_M \eta^2 \operatorname{div} (e^{-\frac{|\vec{x}|^2}{4}} \nabla g) + \int_M \eta^2 (|\nabla g|^2 + |B|^2) e^{-\frac{|\vec{x}|^2}{4}} \\ &= - \int_M 2\eta \langle \nabla g, \nabla \eta \rangle e^{-\frac{|\vec{x}|^2}{4}} + \int_M \eta^2 (|\nabla g|^2 + |B|^2) e^{-\frac{|\vec{x}|^2}{4}} \\ &\geq - \int_M (\eta^2 |\nabla g|^2 + |\nabla \eta|^2) e^{-\frac{|\vec{x}|^2}{4}} + \int_M \eta^2 (|\nabla g|^2 + |B|^2) e^{-\frac{|\vec{x}|^2}{4}} \\ &= \int_M (-|\nabla \eta|^2 + \eta^2 |B|^2) e^{-\frac{|\vec{x}|^2}{4}}, \end{aligned}$$

and the conclusion follows.  $\square$

**Proof of theorem 1.1:** Because  $M$  is a graphic submanifold of  $R^{n+m}$ , we can find an  $n$ -vector  $A$  such that  $\omega$  is everywhere positive on  $M$ . Let  $D_r = M \cap B_r$ , where  $B_r$  is the ball in  $R^{n+m}$  centered at the origin with radius  $r > 1$ , and we choose  $0 \leq \eta \leq 1$  to be a function defined on  $M$  which equals to 1 on  $D_r$  and equals to zero outside  $D_{r+1}$ , with first derivatives bounded by a constant  $C$  independent of  $r$ . Then by lemma 3.1 we have

$$\int_{D_r} |B|^2 e^{-\frac{|\vec{x}|^2}{4}} \leq \int_M \eta^2 |B|^2 e^{-\frac{|\vec{x}|^2}{4}} \leq \int_M |\nabla \eta|^2 e^{-\frac{|\vec{x}|^2}{4}} \leq C \int_{D_{r+1} \setminus D_r} e^{-\frac{|\vec{x}|^2}{4}}.$$

Let  $r \rightarrow \infty$  and by theorem 2.1 (note that  $M$  is a graph in  $R^{n+m}$ , so it is proper) we have  $B \equiv 0$ , which completes the proof of theorem 1.1.

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